§3.5 Series

Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Consider the sequence $(s_n)_{n\in\mathbb{N}}$ of the partial sums

$$s_n = a_1 + \dots + a_n = \sum_{k=1}^{n} a_k, \quad n\in\mathbb{N}.$$

Definition 3.7:

We say, the "series" $\sum_{k=1}^{\infty} a_k$ is "convergent", if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k =: \sum_{k=1}^{\infty} a_k$ exists.

Example 3.7:

i) For $|q|<1$ we have

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

$\Rightarrow$ the "geometric series" is convergent

Proof:

Let $0<q<1$. Then for each $n\in\mathbb{N}$

$$s_n := 1+q+\dots+q^n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

Proof by induction:

a) $n=0$ ✓, $n=1$ ✓

b) $n \mapsto n+1$:

Assume the claim holds for n. Then

$$S_{n+1} = S_n + q^{n+1} = \frac{1-q^{n+1}}{1-q} + q^{n+1}$$

$$= \frac{1-q^{n+1} + (1-q)q^{n+1}}{1-q} = \frac{1-q^{n+2}}{1-q} \quad \checkmark$$

According to Example 3.1 ii) $\lim_{n\to\infty} q^n = 0$

Thus

$$\lim_{n\to\infty} S_n = \frac{1}{1-q}$$

$\square$

ii) The "harmonic series" of Example 3.4 is divergent.

iii) The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent

Proof:

We have to show that the sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ has a limit

We simplify $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

$$\Longrightarrow s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

and so $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$\square$

Convergence criteria:

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with

$$S_n = \sum_{k=1}^{n} a_n \, , \quad n \in \mathbb{N}$$

Then, Prop. 3.4 and 3.5 give:

Proposition 3.9: (Cauchy criterion)

The series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if

$$\left| \sum_{k=\ell}^{n} a_k \right| \longrightarrow 0 \;(n \geq \ell \longrightarrow \infty)$$

Proof:

$|S_n - S_\ell| = \left| \sum_{k=\ell+1}^{n} a_k \right|$. Use propositions 3.4 and 3.5.

$\square$

Remark 3.3:

i) In particular, the condition $a_k \longrightarrow 0 \;(k\to\infty)$ is "necessary" for convergence of

$\sum_{k=1}^{\infty} a_k$ (Choose $n=\ell$ in Prop. 3.9)

ii) The condition $a_k \to 0$ $(k \to \infty)$ is not "sufficient" for convergence (e.g. harmonic series)

<u>Proposition 3.10</u> (Quotient criterion):

Let $a_k \neq 0$, $k \in \mathbb{N}$.

i) If

$$\limsup_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1,$$

then $\sum_{k=1}^{\infty} a_k$ is convergent.

ii) If

$$\liminf \left| \frac{a_{k+1}}{a_k} \right| > 1,$$

then $\sum_{k=1}^{\infty} a_k$ is divergent.

<u>Proof:</u>

i) Set $q_0 := \limsup_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{n \to \infty} \sup_{k \geq n} \left| \frac{a_{k+1}}{a_k} \right| < 1$

Choose $q \in \mathbb{R}$ s.t. $q_0 < q < 1$. Then $\exists\, n_0 \in \mathbb{N}$ with $\forall\, n \geq n_0$: $\sup_{k \geq n} \left| \frac{a_{k+1}}{a_k} \right| \leq q$, and in particular for $n = n_0$ also

$$\forall \; k \geq n_0 : \left| \frac{a_{k+1}}{a_k} \right| \leq q .$$

Then
$$|a_k| = \left| \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_{n_0+1}}{a_{n_0}} a_{n_0} \right|$$

$$\leq \underbrace{q^{-n_0} |a_{n_0}|}_{=: C} q^k$$

$$\Rightarrow \left| \sum_{k=\ell}^{n} a_k \right| \leq \sum_{k=\ell}^{n} |a_k| \leq C \sum_{k=\ell}^{n} q^k \leq C q^{\ell} \frac{1}{1-q}$$

$$\left(\text{indeed } \sum_{k=\ell}^{n} q^k \leq \sum_{k=0}^{\infty} q^k - \sum_{k=0}^{\ell-1} q^k = \frac{1}{1-q} - \frac{1-q^{\ell}}{1-q} = q^{\ell} \frac{1}{1-q} \right)$$

$$\Rightarrow \left| \sum_{k=\ell}^{n} a_k \right| \leq C q^{\ell} \frac{1}{1-q} \longrightarrow 0 \; (n \geq \ell \to \infty)$$

Prop. 3.9 $\Rightarrow \sum_{k=1}^{\infty} a_k$ is convergent

ii) analogous

$\square$

<u>Example 3.8:</u>

i) The series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges

<u>Proof:</u>

set $a_n := \frac{n^2}{2^n}$ and we have for $n \geq 3$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{1}{2} \left(1 + \frac{1}{n} \right)^2$$

$$\leq \frac{1}{2}\left(1+\frac{1}{3}\right)^2 = \frac{8}{9} =: q_0 < 1,$$

Prop. 3.10 $\Rightarrow$ the quotient criterion is satisfied

$\square$

ii) For $a_n := \frac{1}{n^2}$ we obtain the converging sequence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Proof:

We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2} < 1 \quad \forall \ n \geq 1$$

However: there is no $q_0 < 1$ with

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q_0 \quad \forall \ n \geq n_0$$

$\Rightarrow$ We are not allowed to use the quotient criterion!

Use different reasoning for the proof:

According to Example 3.7 iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{2}{n(n+1)}$ also converges since $\forall \ n \geq 1$: $\frac{1}{n^2} \leq \frac{2}{n(n+1)}$, we thus have:

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 2$$

$\Rightarrow S_n$ is monotonically increasing and bounded!

Prop. 3.8 $\Rightarrow$ convergent $\square$

§4. Continuity

§4.1 Limits of Functions

So far we can compute limits of expressions like $y_k = \dfrac{a_k b_k + c_k}{d_k}$

with $a_k \to a$, $b_k \to b$, $c_k \to c$, $d_k \to d$ $(k \to \infty)$

More generally, we can determine for a function $f: D \to \mathbb{R}$ on $D \subset \mathbb{R}$ the convergence of the sequence $y_k = f(x_k)$, where $(x_k)_{k\in\mathbb{N}}$ with $x_k \to x_0$ $(k \to \infty)$.

The limit x_0 of the sequence $(x_k)_{k\in\mathbb{N}}$ doesn't have to be inside D.

Example 4.1:

Let $f(x) = \dfrac{x^2-1}{x-1}$, $x \neq 1$. The function f has a "singularity" at $x=1$. Then

$$x^2-1 = (x+1)(x-1) \Longrightarrow f(x) = x+1 \text{ for } x \neq 1$$

Therefore, for a sequence $1 \neq x_k \to x_0 := 1$ for $k \to \infty$ we obtain $f(x_k) = x_k + 1 \to 2$
$(k \to \infty)$

Definition 4.1:

For a set $D \subset \mathbb{R}$, we define its "closure" $\overline{D}$ as the set of all points in D as well as all limit points of D (limits of sequences in D).

Let $f: D \longrightarrow \mathbb{R}$, $x_0 \in \overline{D}$.

Definition 4.2:

f has a limit $a \in \mathbb{R}$ at x_0, if for each sequence $(x_k)_{k \in \mathbb{N}}$ in D with $x_k \longrightarrow x_0$ $(k \to \infty)$ we have $f(x_k) \longrightarrow a$ $(k \longrightarrow \infty)$

Notation: $\lim_{x \longrightarrow x_0} f(x) = a$

Example 4.2:

For the function f in Example 4.1, we have $\lim_{x \longrightarrow 1} f(x) = 2$

Remark 4.1:

If $\lim_{x \longrightarrow x_0} f(x) =: a$ exists, and if further $x_0 \in D$, then $a = f(x_0)$. (Consider the constant sequence $x_k = x_0 \in D$, $k \in \mathbb{N}$)

Definition 4.3:

Let $D \subset \mathbb{R}$, $f: D \to \mathbb{R}$.

i) f is called "continuous at" $x_0 \in D$, if $\lim_{x \to x_0} f(x) =: a$ exists (given by $a = f(x_0)$).

ii) x_0 is called a "removable singularity", if $x_0 \in \overline{D} \setminus D$ and $\lim_{x \to x_0} f(x) =: a$ exists.
(In this case the completion of the function f by $f(x_0) = a$ is continuous at x_0).

Example 4.3:

1) Let $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$. As shown in Example 4.1, we can complete the function f at $x=1$ through $f(1)=2$ in a continuous way.

ii) Let $D = \mathbb{R} \setminus \{0\}$, $f(x) = \frac{1}{x}$, $x \neq 0$. Then we have for $x_k \to x_0 \neq 0$ according to Prop. 3.3: $f(x_k) \to \frac{1}{x_0}$ $(k \to \infty)$

At $x_0 = 0$ the function f does <u>not</u> have a limit. Consider for example $x_k = \frac{1}{k} \to 0$ $(k \to \infty)$

with $f(x_k) = k \to \infty \ (k \to \infty)$.

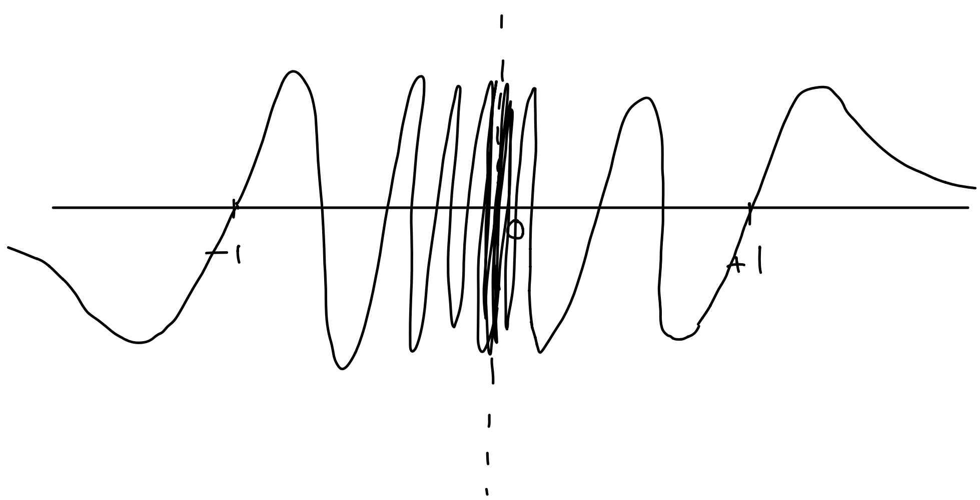
iii) Let $f(x) = \sin\left(\frac{\pi}{x}\right)$. Then we have

$$f(1) = \sin \pi = 0, \quad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0,$$

$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0, \quad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$

Similarly, $f(0.01) = f(0.001) = f(0.0001) = 0$

But $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ does <u>not</u> exist



iv) The piecewise constant function

$f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ with

$$f(x) = \begin{cases} a, & x < 0 \\ b, & x > 0 \end{cases}$$

is continuous at every $x_0 \neq 0$. But for $a \neq b$ there exists no continuous completion at $x_0 = 0$

§4.2 Continuity criteria

Proposition 4.1:

Let $f: D \to \mathbb{R}$, $x_0 \in D$. Then the following are equivalent

i) (sequence criterion) f is continuous at x_0 according to Definition 4.3

ii) (Weierstrass ε-δ criterion):

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D:$$

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

iii) For each interval $V \subset \mathbb{R}$ with $f(x_0) \in V$, we have that $U = f^{-1}(V)$ is an interval in D containing x_0.

Proof:

i) $\implies$ ii):

Let $(x_n)_{n\in\mathbb{N}} \subset D$ be a sequence with $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Assume ii) does not hold. Then there exists $\varepsilon > 0$, s.t. there is no $\delta > 0$ with

$$|f(x) - f(x_0)| < \varepsilon \quad \forall \ x \in D \text{ with } |x - x_0| < \delta$$

$\implies \exists \ x \in D$ with $|x - x_0| < \delta$, but $|f(x) - f(x_0)| \geq \varepsilon$

Thus for every natural number $n \geq 1$, there exists $x_n \in D$ with:

$$|x_n - x_0| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(x_0)| \geq \varepsilon \qquad (*)$$

$$\Rightarrow \lim_{n\to\infty} x_n = x_0 \xrightarrow{f \text{ is continuous}} f(x_n) = f(x_0)$$

But this is in contradiction to $(*)$

ii) $\Rightarrow$ i) : Assume ii) holds. Then we have to show that for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in D$ and $\lim\limits_{n\to\infty} x_n = x_0$ we have

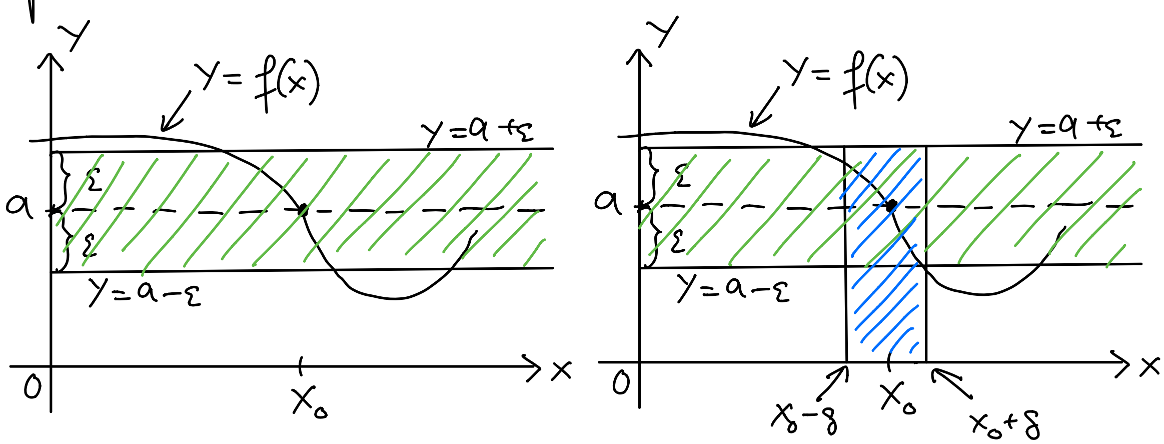
$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

Let $\varepsilon > 0$ and let $\delta > 0$ be given according to ii). As $\lim\limits_{n\to\infty} x_n = x_0$, there exists $n_0 \in \mathbb{N}$, with $|x_n - x_0| < \delta \ \forall \ n \geq n_0$.

$$\Rightarrow |f(x_n) - f(x_0)| < \varepsilon \quad \forall \ n \geq n_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$$

$\square$

picture:

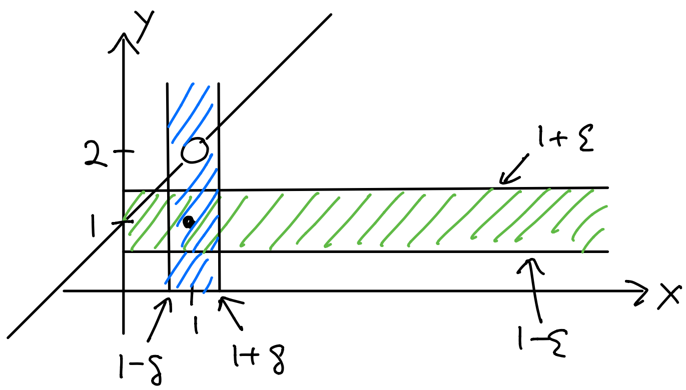


Example 4.4:

i) The function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} &, \text{ if } x \neq 1 \\ 1 &, \text{ if } x = 1 \end{cases}$$

is not continuous at $x = 1$



ii) Let $D = \mathbb{R}$, $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be the "characteristic function" of $\mathbb{Q}$ with

$$f(x) := \chi_{\mathbb{Q}} = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Then $f(x)$ is discontinuous everywhere on $\mathbb{Q}$ as $f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = \mathbb{Q}$ and $\mathbb{Q}$ is not an interval containing x_0.